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On the convergence of double Elzaki transform

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A B S T R A C T

In this research, we have studied the convergence properties of Double Elzaki transformation and the results have been presented in the form of theorems on convergence, absolute convergence and uniform convergence of Double Elzaki transformation. The Double Elzaki transform of double Integral has also been discussed for integral evaluation. Finally, we have solved a Volterra integro-partial differential equation by using Double Elzaki transformation.

1. Introduction

Integral transforms are valuable for the simplification that they bring about, most often in dealing with differential equation subject to particular boundary conditions. Proper choice of the class of transformation usually makes it possible to convert not only the derivatives in an intractable differential equation but also the boundary values into terms of an algebraic equation that can be easily solved. The solution obtained is, of course, the transform of the solution of the original differential equation, and it is necessary to invert this transform to complete the operation.

Integral transform, mathematical operator that produces a new function $f(y)$ by integrating the product of an existing function $F(x)$ and a so-called kernel function $K(x,y)$ between suitable limits. The process, which is called transformation, is symbolized by the equation $f(y) = \int K(x,y)F(x)dx$. Several transforms are commonly named for the mathematicians who introduced them. In the Laplace transform, the kernel is e^{-xy} and the limits of integration are zero and plus infinity, in the Fourier transform, the kernel is $(2\pi)^{-1/2}\, e^{-ixy}$ and the limits are minus and plus infinity. In Schiff (2013), The Laplace transform of f as

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$$
$$= \lim_{n\to\infty} \int_0^\tau e^{-st} f(t)dt$$

whenever the limit exists (as a finite number).

When it does, the above integral is said to converge. If the limit does not exist, the integral is said to diverge and there is no Laplace transform defined for *f*. The notation $L(f)$ will also be used to denote the Laplace transform of *f*, and the integral is the ordinary Riemann (improper) integral. The parameter s belongs to some domain on the real line or in the complex plane.

In Belgacem et al. (2003) and Watugala (1993), a new integral transform, called the Sumudu transform defined for functions of exponential order. We consider functions in the set A, defined by

$$A = \{f(t)\,|\exists M, \tau_1, and\ /or\ \tau_2 > 0,$$
$$such\ that\,|\,f(t)| < Me^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty)\}$$

For a given function in the set A, the constant M must be finite, while τ_1 and τ_2 need not simultaneously exist, and each may be infinite. Instead of being used as a power to the exponential as in the case of the Laplace transform, the variable u in the Sumudu transform is used to factor the variable t in the argument of the function *f*. Specifically, for $f(t)$ in A, the Sumudu transform is defined by

$$G(u) = s[f(t)] = \begin{cases} \int_0^\infty f(ut)e^{-t}dt, & 0 \le u < \tau_2 \\ \int_0^\infty f(ut)e^{-t}dt, & -\tau_1 \le u < 0 \end{cases}$$

Belgacem (2007) presented the fundamental properties, analytical investigation of the samudu transform and applications to integral equations. In Belgacem (2007) and Belgacem and Karaballi (2006), all existing samudu shifting theorems and recurrence results have been generalized, also presented applications to convolution type integral equations with focus on production problems and

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inverse Sumudu transform of a singular function satisfies the Tauberian theorem, where the dirac delta function fails.

In Belgacem (2006, 2009), Laplace transform definition is implemented without resorting to Adomian decomposition nor Homotopy perturbation methods. He also applied the natural transform to Maxwell's equations and obtained the transient electric and magnetic field solution. In Belgacem and Silambarasan (2017b), The Samudu transform is applied to arbitrary powers Dumont bimodular Jacobi elliptic functions for arbitrary powers. Belgacem (2010) applied the Samudu transform applications to Bessel's functions and equations. In Belgacem and Silambarasan (2017a), the Samudu transform integral equation is solved by continuous integration by parts, to obtain its definition for trigonometric functions.

Belgacem and Al-Shemas (2014) proposed ideas towards the mathematical investigations of the environmental fitness effects on populations dispersal and persistence. Goswami and Belgacem (2012) giave a sufficient condition to guarantee the solution of the constant coefficient fractional differential equations by Samudu transform.

The Elzaki Transform is a new integral transform introduced by Elzaki (2011a). Elzaki Transform is modified form of Laplace Transform. Elzaki transform is well applied to initial value problems with variable coefficients and solving integral equations of convolution type. Elzaki Transform is also used to find solution of system of partial differential equations (Elzaki, 2011b). Typically, Fourier, Laplace and Sumudu transforms are the convenient mathematical tools for solving differential equations.

Elzaki Transformation is defined for the function of exponential order. Consider a function in the set S defined as

$$S=\left\{f(t): \exists\, M, k_1, k_2>0, \left|\, f(t)\right|<Me^{\frac{|t|}{k_j}}, if\ t\in (-1)^j\times[0,\infty)\right\}$$

For a given function f(t) in the set S, the constant M must be finite, number k_1, k_2 may be finite or infinite.

The Elzaki Transform denoted by the operator E is defined as

$$E\,[f(t)] = T\,(v) = v\int_0^\infty f(t)e^{-\frac{t}{v}}dt \quad ,t\geq 0$$

The variable v in this transform is used to factorize the variable t in the argument of the function f. The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving some convergence theorems.

2. Convergence theorem of double Elzaki integral

In this section, we prove the convergence theorem of double Elzaki integral.

Theorem 2.1: Let $\phi(x,t)$ be a function of two variables continuous in the positive quard of the *xt*-plane. If the integral converges at $p=p_o$ *and* $s=s_o$ then integral converges for $p<p_o, s<s_o$.

$$ps\int_0^\infty\int_0^\infty e^{-\frac{x}{p}-\frac{t}{s}}\,\phi(x,t)dxdt \tag{1}$$

for the proof we will use the following lemmas.

Lemma 2.2: If the integral

$$s\int_0^\infty e^{-\frac{t}{s}}\,\phi(x,t)dt \tag{2}$$

converges at s= s_o then the integral converges for $s<s_o$

Proof: Consider the set

$$\alpha(x,t) = s_o\int_0^t e^{-\frac{u}{s_o}}\,\phi(x,u)du \;,(o<t<\infty). \tag{3}$$

clearly, $\alpha(x,0)=0$ and $\lim_{t\to\infty}\alpha(x,t)$ exist because $s\int_0^\infty e^{-\frac{t}{s}}\,\phi(x,t)dt$ converges at $s=s_o$.

By fundamental theorem of Calculus we have

$$\alpha_t\,(x,t) = s_o e^{-\frac{t}{s}}\phi(x,t).$$

If we choose ϵ_1 and R_1 such that $(0<\epsilon_1<R_1)$ then

$$s\int_{\epsilon_1}^{R_1} e^{-\frac{t}{s}}\,\phi(x,t)dt = s\int_{\epsilon_1}^{R_1} e^{-\frac{t}{s}}\frac{1}{s_o}\alpha_t(x,t)e^{\frac{t}{s_o}}\,dt$$
$$=\frac{s}{s_o}\int_{\epsilon_1}^{R_1} e^{-\left(\frac{s_o-s}{ss_o}\right)t}\;\alpha_t(x,t)\,dt$$

By using integration by parts

$$=\frac{s}{s_o}\left\{\left|e^{-\left(\frac{s_o-s}{ss_o}\right)t}\;\alpha(x,t)\right|_{\epsilon_1}^{R_1} - \int_{\epsilon_1}^{R_1}\alpha(x,t)e^{-\left(\frac{s_o-s}{ss_o}\right)t}\left[-\left(\frac{s_o-s}{ss_o}\right)\right]dt\right\}$$
$$=\frac{s}{s_o}\left[e^{-\left(\frac{s_o-s}{ss_o}\right)R_1}\;\alpha(x,R_1) - e^{-\left(\frac{s_o-s}{ss_o}\right)\epsilon_1}\;\alpha(x,\epsilon_1)+\left(\frac{s_o-s}{ss_o}\right)\int_{\epsilon_1}^{R_1}\alpha(x,t)e^{-\left(\frac{s_o-s}{ss_o}\right)t}\,dt\,\right].$$

Now let $\epsilon_1\to 0$. Both terms on the right which depend on ϵ_1 approach a limit and

$$s\int_0^{R_1} e^{-\frac{t}{s}}\,\phi(x,t)dt = \frac{s}{s_o}\left[\,e^{-\left(\frac{s_o-s}{ss_o}\right)R_1}\;\alpha(x,R_1)+\left(\frac{s_o-s}{ss_o}\right)\int_0^{R_1}\alpha(x,t)e^{-\left(\frac{s_o-s}{ss_o}\right)t}\,dt\right]$$

Now let $R_1\to\infty$. If $s<s_o$, the first term on the right approaches zero and

$$s\int_0^\infty e^{-\frac{t}{s}}\,\phi(x,t)dt = \left(\frac{s_o-s}{{s_o}^2}\right)\int_0^\infty e^{-\left(\frac{s_o-s}{ss_o}\right)t}\;\alpha(x,t)\,dt \qquad \text{for } s<s_o \tag{4}$$

The given theorem is proved if the integral on the right converges.

Now by using the "Limit test" for convergence (Widder, 2005). For this we have

$$\lim_{t\to\infty} t^2 e^{-\left(\frac{s_o-s}{ss_o}\right)t}\alpha(x,t) = \left[\lim_{t\to\infty}\frac{t^2}{e^{\left(\frac{s_o-s}{ss_o}\right)t}}\right]\left[\lim_{t\to\infty}\alpha(x,t)\right]$$
$$= 0 * \left[\lim_{t\to\infty}\alpha(x,t)\right] = 0 = \text{finite}$$

Therefore, the integral on right hand side of (4) converges for $s < s_o$.

Hence the given integral $s\int_0^\infty e^{-\frac{t}{s}}\,\phi(x,t)dt$ converges for $s < s_o$.

Lemma 2.3: If integral

$$h(x,s) = s\int_0^\infty e^{-\frac{t}{s}}\,\phi(x,t)dt \tag{5}$$

converges for $s \le s_o$ and integral

$$p\int_0^\infty e^{-\frac{x}{p}}\, h(x,s)dx \tag{6}$$

converges at $p = p_o$ then the integral (6) converges for $p < p_o$

Proof:

Let $\beta(x,s) = p_o\int_0^x e^{-\frac{u}{p_o}}\, h(u,s)du \quad 0 < x < \infty \quad (7)$

Therefore $\beta(0,s) = 0$ and $\lim_{x\to\infty}\beta(x,s)$ exist because $p\int_0^\infty e^{-\frac{x}{p}}\, h(x,s)dt$ converges at $p = p_o$.

By using fundamental theorem of Calculus, equation (7) can be written as $\beta_x(x,s) = p_o e^{-\frac{x}{p_o}} h(x,s)$. Choose ϵ_2 and R_2 and so that $0 < \epsilon_2 < R_2$

$$p\int_{\epsilon_2}^{R_2} e^{-\frac{x}{p}}\, h(x,s)dx = \frac{p}{p_o}\int_{\epsilon_2}^{R_2} e^{-\left(\frac{p_o-p}{pp_o}\right)x}\beta_x(x,s)dx$$
$$= \frac{p}{p_o}\Big[e^{-\left(\frac{p_o-p}{pp_o}\right)R_2}\beta_x(R_2,s) - e^{-\left(\frac{p_o-p}{pp_o}\right)\epsilon_2}\beta_x(\epsilon_2,s) + \left(\frac{p_o-p}{pp_o}\right)\int_{\epsilon_2}^{R_2} e^{-\left(\frac{p_o-p}{pp_o}\right)x}\beta(x,s)\,dx\Big]$$

Now let $\epsilon_2 \to 0$. Both terms on the right which depend on ϵ_2 approach a limit and

$$p\int_0^{R_2} e^{-\frac{x}{p}}\, h(x,s)dx = \frac{p}{p_o}\left[e^{-\left(\frac{p_o-p}{pp_o}\right)R_2}\beta_x(R_2,s) + \left(\frac{p_o-p}{pp_o}\right)\int_0^{R_2} e^{-\left(\frac{p_o-p}{pp_o}\right)x}\beta(x,s)\,dx\right] \tag{8}$$

Now let $R_2 \to \infty$. If $s < s_o$, the first term on the right approaches zero.

$$p\int_0^\infty e^{-\frac{x}{p}}\, h(x,s)dx = \left(\frac{p_o-p}{{p_o}^2}\right)\int_0^\infty e^{-\left(\frac{p_o-p}{pp_o}\right)x}\beta(x,s)\,dx \quad \text{for } p < p_o \tag{9}$$

The given theorem is proved if the integral on the right converges.

Now by using the Limit test for convergence (Widder, 2005). We consider

$$\lim_{x\to\infty} x^2 e^{-\left(\frac{p_o-p}{pp_o}\right)x}\beta(x,t) = \left[\lim_{x\to\infty}\frac{x^2}{e^{-\left(\frac{p_o-p}{pp_o}\right)x}}\right]\left[\lim_{x\to\infty}\beta(x,t)\right]$$
$$= 0 * \left[\lim_{x\to\infty}\beta(x,t)\right] = 0 = \text{finite}$$

Therefore, the integral on right hand side of (6) converges for $p < p_o$

Hence the given integral $p\int_0^\infty e^{-\frac{x}{p}}\, h(x,s)dx$ converges for $p < p_o$.

The proof of the Theorem 2.1 is as follows

$$ps\int_0^\infty\int_0^\infty e^{-\frac{x}{p}-\frac{t}{s}}\,\phi(x,t)dxdt =$$
$$p\int_0^\infty e^{-\frac{x}{p}}\{s\int_0^\infty e^{-\frac{t}{s}}\,\phi(x,t)dt\}dx$$
$$= p\int_0^\infty e^{-\frac{x}{p}} h(x,t)dx \tag{10}$$

where

$$h(x,s) = s\int_0^\infty e^{-\frac{t}{s}}\phi(x,t)dt$$

by Lemma 2.2, integral $s\int_0^\infty e^{-\frac{t}{s}}\phi(x,t)dt$ converges for $s < s_o$.

Also by Lemma 2.3, integral $p\int_0^\infty e^{-\frac{x}{p}}\, h(x,s)dx$ converges for $p < p_o$.

Therefore, the integral in RHS of (10) converges for $p < p_o, s < s_o$. Hence the integral

$$ps\int_0^\infty\int_0^\infty e^{-\frac{x}{p}-\frac{t}{s}}\,\phi(x,t)dxdt$$

converges for $p < p_o, s < s_o$. This completes the proof of the Theorem 2.1

Corollary 2.4: If the integral (1) diverges at $p = p_o$ and $s = s_o$ then the integral (1) diverges at $p < p_o$, $s < s_o$.

Corollary 2.5: The region of the convergence of the integral (1) is the positive quadrant of the *xt*-plane. Now we prove absolute convergence of integral (1).

Theorem 2.6: If the integral (1) converges absolutely at $p = p_o$, $s = s_o$ then integral (1) converges absolutely for $p \le p_o$, $s \le s_o$.

Proof: We know that

$e^{-\frac{x}{p_o}-\frac{t}{s_o}}\,|\,\phi(x,t)\,| \le e^{-\frac{x}{p}-\frac{t}{s}}$ for $(p \le p_o < \infty, s \le s_o < \infty)$;

therefore

$$s_o p_o\int_0^\infty\int_0^\infty e^{-\frac{x}{p_o}-\frac{t}{s_o}}\,|\,\phi(x,t)\,|dtdx \le$$
$$ps\int_0^\infty\int_0^\infty e^{-\frac{x}{p}-\frac{t}{s}}\,|\,\phi(x,t)\,|dtdx;$$

form given hypothesis,

$$ps\int_0^\infty\int_0^\infty e^{-\frac{x}{p}-\frac{t}{s}}\,|\,\phi(x,t)\,|dtdx;$$

converge. hence, we have

$$s_o p_o\int_0^\infty\int_0^\infty e^{-\frac{x}{p_o}-\frac{t}{s_o}}\,|\,\phi(x,t)\,|dtdx$$

converge for $(p \le p_o\ s \le s_o)$.

Therefore the integral (1) converges absolutely for ($p \le p_o$ $s \le s_o$).

3. Uniform convergence

In this section we prove the uniform convergence of double Elzaki Transform.

Theorem 3.1: If *f (x, t)* is continuous on $[0,\infty)\times[0,\infty)$ and

$$H(x,t) = p_o s_o \int_0^x \int_0^t e^{-\frac{u}{p_o}-\frac{v}{s_o}} f(u,v)\,du\,dv \qquad (11)$$

is bounded on $[0,\infty)\times[0,\infty)$, then the double of Elzaki Transform of f converges uniformly on $[p,\infty)\times[s,\infty)$ if $p < p_o,\ s < s_o$. For the proof we will use the following lemmas:

Lemma 3.2: If $g(x,t) = s_o \int_0^t e^{-\frac{v}{s_o}} f(x,v)dv$ is bounded on $[0,\infty)$ then the Elzaki Transform of f with respect to s converges uniformly on $[s,\infty)$ if $s < s_o$.

Proof: If $0 \le r \le r_1$ then consider

$$s\int_r^{r_1} e^{-\frac{t}{s}} f(x,t)dt = s\int_r^{r_1} e^{-\left(\frac{s_o-s}{ss_o}\right)t} e^{-\frac{t}{s_o}} f(x,t)dt$$
$$= \frac{s}{s_o}\int_r^{r_1} e^{-\left(\frac{s_o-s}{ss_o}\right)t} g_t(x,t)dt$$

Using integration by parts

$$= \frac{s}{s_o}[e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} g(x,r_1) - e^{-\left(\frac{s_o-s}{ss_o}\right)r} g(x,r) + \left(\frac{s_o-s}{ss_o}\right)\int_r^{r_1} e^{-\left(\frac{s_o-s}{ss_o}\right)t} g(x,t)dt]$$

Therefore, if $|g(x,t)| \le M$ then.

$$\left|s\int_r^{r_1} e^{-\frac{t}{s}} f(x,t)dt\right| \le M\left\{e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} + e^{-\left(\frac{s_o-s}{ss_o}\right)r} + \left(\frac{s_o-s}{ss_o}\right)\int_r^{r_1} e^{-\left(\frac{s_o-s}{ss_o}\right)t} dt\right\}$$
$$= M\left\{e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} + e^{-\left(\frac{s_o-s}{ss_o}\right)r} - e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} + e^{-\left(\frac{s_o-s}{ss_o}\right)r}\right\}$$
$$= 2Me^{-\left(\frac{s_o-s}{ss_o}\right)r} \quad for\ , s < s_o$$

By Cauchy criterion for uniform convergence (Trench, 2012).

$$s\int_r^{r_1} e^{-\frac{t}{s}} f(x,t)dt$$

converges uniformly on $[s,\infty)$if $s < s_o$.

Hence, Elzaki Transform of f with respect to s converges uniformly on $[s,\infty)$if $s < s_o$

Lemma 3.3: If the integral $g(x,s) = s\int_0^\infty e^{-\frac{t}{s}} f(x,t)dt$ converges uniformly on $[s,\infty)$if $s < s_o$ and $\alpha(x,s) = p_o\int_0^x e^{-\frac{u}{p_o}} g(u,s)du$ is bounded on $[0,\infty)$ then the Elzaki Transform of f with respect to s converges uniformly on $[p,\infty)$if $p < p_o$

Proof: Proof is similar to Lemma 3.2. The proof of the Theorem 3.1 is as follows:

$$H(x,t) = p_o s_o \int_0^x \int_0^t e^{-\frac{u}{p_o}-\frac{v}{s_o}} f(u,v)dudv$$
$$= p_o\int_0^x e^{-\frac{u}{p_o}}\left\{s_o\int_0^t e^{-\frac{v}{s_o}} f(u,v)dv\right\}du$$
$$= p_o\int_0^x e^{-\frac{u}{p_o}} g(u,t)\,du$$

where $g(u,t) = s_o\int_0^t e^{-\frac{v}{s_o}} f(u,v)dv$ is bounded on $[0,\infty)$.

By Lemma 3.2, Elzaki transform of f with respect to s converges uniformly on $[s,\infty)$ if $s < s_o$.

Also by Lemma 3.3, Elzaki transform of g with respect to p converges uniformly on $[p,\infty)$ if $p < p_o$.

Hence double Elzaki transform of f converges uniformly on $[p,\infty)\times[s,\infty)$ if $p < p_o,\ s < s_o$

We now prove the differentiability of double Elzaki transform

Theorem 3.4: If *f (x, t)* is continuous on $[0,\infty)\times[0,\infty)$ and

$$H(x,t) = p_o s_o \int_0^x \int_0^t e^{-\frac{u}{p_o}-\frac{v}{s_o}} f(u,v)dudv$$

Is bounded on $[0,\infty)\times[0,\infty)$ then the double Elzaki transform of f is infinitely differentiable with respect to p and s on $[p,\infty)\times[s,\infty)$if $p < p_o$, $s < s_o$, with

$$\frac{\partial^{m+n}}{\partial p^m \partial s^n}\bar{\bar{f}}(p,s) = (-1)^{m+n}\ ps\int_0^\infty\int_0^\infty e^{-\frac{x}{p}-\frac{t}{s}} x^m t^n f(x,t)dx\,dt \qquad (12)$$

for the proof we will use the following lemmas

Lemma 3.5: If $g(x,t) = s_o\int_0^t e^{-\frac{v}{s_o}} f(x,v)dv$ is bounded on $[0,\infty)$ then the Elzaki transform of f is infinitely differentiable with respect to s on $[s,\infty)$ if $s < s_o$ with

$$\frac{\partial^n}{\partial s^n}\bar{f}(x,s) = (-1)^n s\int_0^\infty e^{-\frac{t}{s}} t^n f(x,t)dt \qquad (13)$$

Proof: First of all we will prove that the integrals

$$I_n(x,s) = (-1)^n s\int_0^\infty e^{-\frac{t}{s}} t^n f(x,t)dt \quad ,n = 0,1,2,3,\dots$$

all converge uniformly on $[s,\infty)$ if $s < s_o$ and if $0 \le r \le r_1$, then

$$s\int_r^{r_1} e^{-\frac{t}{s}} t^n f(x,t)dt = \frac{s}{s_o}\int_r^{r_1} e^{-\left(\frac{s_o-s}{ss_o}\right)t} t^n g_t(x,t)dt$$
$$= \frac{s}{s_o}[e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} r_1^n g(x,r_1) - e^{-\left(\frac{s_o-s}{ss_o}\right)r} r^n g(x,r) - \int_r^{r_1}\{\frac{d}{dt}e^{-\left(\frac{s_o-s}{ss_o}\right)t} t^n\} g(x,t)dt]$$

Therefore, if $|g(x,t)| \le M < \infty$ on $[0,\infty)$ then

$\left| s\int_r^{r_1} e^{-\frac{t}{s}} t^n f(x,t)dt \right| \leq M\left\{e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} r_1^n + e^{-\left(\frac{s_o-s}{ss_o}\right)r} r^n - e^{-\left(\frac{s_o-s}{ss_o}\right)r_1} r_1^n + e^{-\left(\frac{s_o-s}{ss_o}\right)r} r^n\right\}$

$\left| s\int_r^{r_1} e^{-\frac{t}{s}} t^n f(x,t)dt \right| \leq 2Me^{-\left(\frac{s_o-s}{ss_o}\right)r} r^n \ for\ 0 \leq r \leq r_1$

By Cauchy criterion for uniform convergence (Trench, 2012). $I_n(x,s)$ converges uniformly on $[s,\infty)$if $s < s_o$. Now, using Trench (2012) and induction proof, we have (13). That is Elzaki transform of f is infinitely differentiable with respect to s on $[s,\infty)$if $s < s_o$

Lemma 3.6: If the integral $\emptyset(x,s) = s\int_0^\infty e^{-\frac{t}{s}} t^n f(x,t)dt$ converges uniformly on $[s,\infty)$if $s < s_o$ and $h(x,s) = p_o\int_0^x e^{-\frac{u}{p_o}}\emptyset(x,s)dx$ is bounded on $[0,\infty)$ then the Elzaki transform of $\emptyset$ is infinitely differentiable with respect to p on p, ∞) if $p < p_o$, with the

$$\frac{\partial^m}{\partial s^m}\emptyset(x,s) = (-1)^m\, s\int_0^\infty e^{-\frac{t}{s}} t^m\emptyset(x,s)dx \tag{14}$$

Proof: Proof is similar to Lemma 3.5. The proof of the Theorem 3.4 is as follows

$$H(x,t) = p_o s_o\int_0^x\int_0^t e^{-\frac{u}{p_o}-\frac{v}{s_o}} f(u,v)dudv$$
$$= p_o\int_o^x e^{-\frac{u}{p_o}}\{s_o\int_0^t e^{-\frac{v}{s_o}} f(u,v)dv\}du$$
$$= p_o\int_o^x e^{-\frac{u}{p_o}} g(u,t)du$$

where $g(u,t) = s_o\int_0^t e^{-\frac{v}{s_o}} f(u,v)dv$ is bounded on $[0,\infty)$.

By Lemma 3.5, Elzaki transform of f is infinitely differentiable with respect to s on $[s,\infty)$if $s < s_o$.

Also by Lemma 3.6, Elzaki transform of g is infinitely differentiable with respect to p on$[p,\infty)$if $p < p_o$.

Hence double Elzaki transform of f is infinitely differentiable with respect to p and s on $[p,\infty)\times[s,\infty)$ if $s < s_o, p < p_o$.

4. Double Elzaki transform of double integral

We now find the double Elzaki transform of double integral.

Theorem 4.1: If $E_X E_T\{f(x,t)\} = \bar{f}(p,s)$ and

$$g(x,t) = \int_0^x\int_0^t f(u,v)dvdu \tag{15}$$

Then,

$$E_X E_T\left\{\int_0^x\int_0^t f(u,v)dvdu\right\} = ps\,\bar{f}\,(p,s) \tag{16}$$

Proof: If we denote $h(x,t) = \int_0^t f(x,v)dv$. By using fundamental theorem of calculous.

$$h_t(x,t) = f(x,t) \tag{17}$$

Since,

$$h(x,0) = 0 \tag{18}$$

taking double Elzaki transform of equation (17), we get

$$\bar{h}\,(p,s) = s\,\bar{f}(p,s) \tag{19}$$

from (15)

$$g(x,t) = \int_0^x h(u,t)\,du$$
$$g_x(x,t) = h(x,t)\ and\ g(0,t) = 0$$
$$\bar{g}(p,s) = p\,\bar{h}(p,s)$$

now by using (19) and (15), we obtain

$$E_X E_T\left\{\int_0^x\int_0^t f(u,v)dvdu\right\} = ps\bar{f}(p,s).$$

5. Application of double Elzaki transform in Volterra integro-partial differential equation

We use the double Elzaki transform to solve the problem which is already solved in Moghadam and Saeedi (2010) using Differential transform method.

Example 5.1: Consider the following Volterra Integro Partial Differential Equation.

$$\frac{\delta u(x,y)}{\delta x} + \frac{\delta u(x,y)}{\delta y} = -1 + e^x + e^y + e^{x+y} + \int_0^x\int_0^y u(r,t)drdt \tag{20}$$

Subject to the initial conditions as:

$$u(x,0) = e^x\ and\ u(0,y) = e^y \tag{21}$$

Applying double Elzaki transform of equation (20) and we get,

$$\frac{1}{p}\bar{u}(p,s) - p\bar{u}(0,s) + \frac{1}{s}\bar{u}(p,s) - s\bar{u}(p,0)$$
$$= -s^2p^2 + \frac{s^2p^2}{1-p} + \frac{s^2p^2}{1-s} + \frac{s^2p^2}{(1-s)(1-p)} + ps\,\bar{u}\,(p,s) \tag{22}$$

The single Elzaki transforms of equation (21)

$$\bar{u}(p,0) = \frac{p^2}{(1-p)}\quad and\ u(0,s) = \frac{s^2}{(1-s)} \tag{23}$$

Substituting (23) in (22) and simplifying and we obtain,

$$\frac{1}{p}\bar{u}\,(p,s) - p\frac{s^2}{1-s} + \frac{1}{s}\bar{u}(p,s) - s\frac{p^2}{1-p}$$
$$= -s^2p^2 + \frac{s^2p^2}{1-s} + \frac{s^2p^2}{1-p} + \frac{s^2p^2}{(1-s)(1-p)} + ps\bar{u}(p,s)$$

so, $\bar{u}(p,s) = \frac{s^2p^2}{(1-s)(1-p)}$

Now by using double inverse Elzaki transform, we obtain solution of (20) as follows,

$$u(x,y) = e^{x+y}.$$

6. Conclusion

We have proved the convergence, absolute convergence and uniform convergence of double

Elzaki transform. Besides these, we obtained double Elzaki transform of double integral and use it to solve Volterra integro-partial differential equation.

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